A REMARK ON VANISHING CYCLES WITH TWO STRATA

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Abstract. Suppose that the critical locus Σ of a complex analytic function f on affine space is, itself, a space with an isolated singular point at the origin $\mathbf{0}$, and that the Milnor number of f restricted to normal slices of $\Sigma - \{\mathbf{0}\}$ is constant. Then, the general theory of perverse sheaves puts severe restrictions on the cohomology of the Milnor fiber of f at $\mathbf{0}$, and even more surprising restrictions on the cohomology of the Milnor fiber of generic hyperplane slices.

1. Settings

Let $\mathcal{U}$ be an open neighborhood of the origin in $\mathbb{C}^{n+1}$, and $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ be a complex analytic function. Let $(X, \mathbf{0})$ denote the germ of the complex analytic hypersurface defined by this function.

The Milnor fiber, $F_{\mathbf{0}}$, of f at the origin has been a fundamental object in the study of the local, ambient topology of $(X, \mathbf{0})$ since the appearance of the foundational work by Milnor in [11]. In [11], Milnor proves, among other things, that, if f has an isolated critical point at $\mathbf{0}$, then the homotopy-type of $F_{\mathbf{0}}$ is that of a finite one-point union, a *bouquet*, of n-spheres, where the number of spheres is given by the *Milnor number*, $\mu_{\mathbf{0}}(f)$.

It is natural to consider the question of what can be said about the homotopy-type, or even cohomology, of $F_{\mathbf{0}}$ in the case where the dimension of the critical locus (at the origin), $s := \dim_{\mathbf{0}} \Sigma f$, is greater than 0.

One of the first general results along these lines was due to M. Kato and Y. Matsumoto in [4] who proved that, in the case the critical locus of the function f at the origin is s, the Milnor fiber of f at the origin is $(n-s-1)$-connected.

Another general, more computational, result was obtained by the first author, in [5], where it is shown that, up to homotopy, the Milnor fiber of f is obtained from the Milnor fiber of a generic hyperplane restriction $f_{|_H}$ by attaching $\left(\Gamma_{f,H} \cdot X\right)_{\mathbf{0}}$ n-cells, where $\left(\Gamma_{f,H} \cdot X\right)_{\mathbf{0}}$ is the intersection number of the relative polar curve $\Gamma_{f,H}$ with the hypersurface X. In fact, the result of [4] can be obtained directly from [5] (see [2]).

A particular case of the main result of [5] is when the polar curve is empty (or, zero, as a cycle), so that the intersection number above is zero, and the Milnor fiber of f and of $f_{|_H}$ have the same homotopy-type: that of a bouquet of $(n-1)$-spheres.

If Σf is smooth and 1-dimensional, it is trivial to show that $\Gamma_{f,H}$ being empty is equivalent to the Milnor number of the isolated critical point of generic transverse hyperplane sections being constant along Σf. In fact, if Σf is 1-dimensional, one can show, using [6], that $\Gamma_{f,H}$ being empty is equivalent to Σf is smooth and the Milnor number of the isolated critical point of generic transverse hyperplane sections being constant along Σf. Thus, constant transverse Milnor number implies the constancy of the cohomology of the Milnor fiber $F_{\mathbf{p}}$ of f at points $\mathbf{p}$ along Σf.

If Σf is smooth, of arbitrary dimension s, then, proceeding inductively from the 1-dimensional case, one obtains that, if the generic s-codimensional transverse slices of f have constant Milnor

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number along Σf, then the reduced cohomology of the Milnor fiber $F_{\mathbf{p}}$, of f at $\mathbf{p}$, is constant along Σf, and is concentrated in the single degree $n-s$.

What if Σf is smooth, of dimension s, and the generic s-codimensional transverse slices of f have constant Milnor number on $\Sigma f - \{\mathbf{0}\}$, but, perhaps, the transverse slice at $\mathbf{0}$ has a different (necessarily higher) Milnor number? If $s \geq 2$, then, it follows from Proposition 1.31 of [9] that, in fact, the Milnor number of the s-codimensional transverse slices of f have constant Milnor number on all of Σf, i.e., there can be no jump in the transverse Milnor numbers at isolated points on a smooth critical locus of dimension at least 2. The remaining case where $s = 1$ was addressed by the authors in [7].

In this brief paper, we address the case where:

(1) $\Sigma f - \{\mathbf{0}\}$ is smooth near $\mathbf{0}$;
(2) $s \geq 3$;
(3) the Milnor number of a transverse slice of codimension s of the hypersurface $f^{-1}(0)$ is constant along $\Sigma f - \{\mathbf{0}\}$ near $\mathbf{0}$; and
(4) the intersection of Σ with a sufficiently small sphere S_ε centered at $\mathbf{0}$ is $(s-2)$-connected.

Under these hypotheses, we have:

Theorem 1.1. *The Milnor fiber $F_{\mathbf{0}}$ of f at $\mathbf{0}$ can have non-zero cohomology only in degrees 0, $n-s$, $n-1$ and n.*

Corollary 1.2. Suppose that $s \geq 4$ and, for a generic hyperplane H, the real link $S_\varepsilon \cap \Sigma \cap H$ of $\Sigma \cap H$ at $\mathbf{0}$ is $(s-3)$-connected. Then, the Milnor fiber F_H of $f_{|_H}$ at $\mathbf{0}$ can have non-zero cohomology only in degrees 0, $n-s$ and $n-1$.

2. An exact sequence

Let $\mathbb{Z}^\bullet_{\mathcal{U}}$ be the constant sheaf on $\mathcal{U}$ with stalks isomorphic to the ring of integers $\mathbb{Z}$. If ϕ_f is the functor of vanishing cycles of f, we know (see, e.g., [3], Theorem 5.2.21) that the complex $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$ is a perverse sheaf (see, e.g., [1] p. 9) on $f^{-1}(0)$. Let $\mathbf{P}^\bullet$ denote the restriction of this sheaf to its support Σ, which is the set of critical points of f inside $f^{-1}(0)$.

We know that, for all $x \in \Sigma$, we have

$$\mathbb{H}^{-k}(\mathbb{B}(x) \cap \Sigma; \mathbf{P}^\bullet) \;\cong\; H^{-k}(\mathbf{P}^\bullet)_x \;\cong\; \widetilde{H}^{n-k}(F_x; \mathbb{Z}),$$

where F_x is the Milnor fiber of f at x and $\mathbb{B}(x)$ is a sufficiently small ball (open or closed, with non-zero radius) of $\mathbb{C}^{n+1}$ centered at x. Let $\mathbb{B}^*(x) = \mathbb{B}(x) - \{x\}$.

Then, we have the exact sequence in hypercohomology:

$$\begin{aligned}\to \mathbb{H}^{-k}(\mathbb{B}(x) \cap \Sigma, \mathbb{B}^*(x) \cap \Sigma; \mathbf{P}^\bullet) \to \mathbb{H}^{-k}(\mathbb{B}(x) \cap \Sigma; \mathbf{P}^\bullet)& \\ \to \mathbb{H}^{-k}(\mathbb{B}^*(x) \cap \Sigma; \mathbf{P}^\bullet) \to \mathbb{H}^{-k+1}(\mathbb{B}(x) \cap \Sigma, \mathbb{B}^*(x) \cap \Sigma; \mathbf{P}^\bullet) &\to\end{aligned}$$

Since $\mathbf{P}^\bullet$ is perverse, using the cosupport condition (see e.g. [1] p. 9):

$$\mathbb{H}^{-k+1}(\mathbb{B}(x) \cap \Sigma, \mathbb{B}^*(x) \cap \Sigma; \mathbf{P}^\bullet) = 0$$

for $-k+1 < 0$. The support condition (see loc. cit.) leads to:

$$H^{k}(\mathbb{B}(x) \cap \Sigma, \mathbf{P}^\bullet) \;\cong\; \widetilde{H}^{n+k}(F_x; \mathbb{Z}) = 0$$

for $k > 0$. Therefore,

$$\widetilde{H}^{n-k}(F_x; \mathbb{Z}) \;\cong\; \mathbb{H}^{-k}(\mathbb{B}(x) \cap \Sigma; \mathbf{P}^\bullet) \;\cong\; \mathbb{H}^{-k}(\mathbb{B}^*(x) \cap \Sigma; \mathbf{P}^\bullet)$$

for $-k+1 < 0$ and:

$$\widetilde{H}^{k}(F_x; \mathbb{Z}) = 0$$

for $k > n$.

3. Topological hypotheses

Throughout the remainder of this paper, we assume, as in the introduction, that:

(1) $s \geq 3$ (and Σf might be singular at $\mathbf{0}$).
(2) There is an open neighborhood $\mathcal{U}$ of the origin $\mathbf{0}$, such that the Milnor number of a transverse slice of codimension s of the hypersurface $f^{-1}(0)$ is constant along the singular set $\Sigma \cap \mathcal{U} (=\Sigma f \cap \mathcal{U})$ of $X \cap \mathcal{U}$ outside of $\mathbf{0}$, and equal to μ.
(3) The intersection of Σ with a sufficiently small sphere S_ε centered at $\mathbf{0}$ is $(s-2)$-connected.

Note that (1) and (3) imply, in particular, that $S_\varepsilon \cap \Sigma$ is simply-connected. Also (2) implies that $(\Sigma - \{0\}) \cap \mathcal{U} = (\Sigma f - \{0\}) \cap \mathcal{U}$ is smooth.

As we discussed in the introduction, without the language of sheaves, the assumption on the constancy of the Milnor number of f, restricted to a normal slice to Σ, is equivalent to saying that our shifted, restricted vanishing cycle complex $\mathbf{P}^\bullet_{|_{\Sigma-\{\mathbf{0}\}}}$ is locally constant, with stalk cohomology $\mathbb{Z}^\mu$ concentrated in degree $-s$. (The technical details of the sheaf result are non-trivial; see Theorem 6.9 of [9] and Corollary 3.14 of [10].) As $\mathbb{B}^*(\mathbf{0}) \cap \Sigma$ is homotopy-equivalent to $S_\varepsilon \cap \Sigma$, which is simply-connected, it follows that $\mathbf{P}^\bullet_{|_{B^*(\mathbf{0})\cap\Sigma}}$ is isomorphic to the shifted constant sheaf $(\mathbb{Z}^\mu)^\bullet_{B^*(\mathbf{0})\cap\Sigma}[s]$.

This implies that

$$\mathbb{H}^{-k}(\mathbb{B}^*(\mathbf{0}) \cap \Sigma; \mathbf{P}^\bullet) \;\cong\; H^{-k+s}(\mathbb{B}^*(0) \cap \Sigma; \mathbb{Z}^\mu) \;\cong\; H^{-k+s}(S_\varepsilon \cap \Sigma; \mathbb{Z}^\mu).$$

Thus, as $S_\varepsilon \cap \Sigma$ is $(s-2)$-connected, we have:

$$\mathbb{H}^{-s}(\mathbb{B}^*(\mathbf{0}) \cap \Sigma; \mathbf{P}^\bullet) \;\cong\; H^0(S_\varepsilon \cap \Sigma; \mathbb{Z}^\mu) \;\cong\; \mathbb{Z}^\mu,$$

and, if $2 \leq k \leq s-1$:

$$\mathbb{H}^{-k}(\mathbb{B}^*(\mathbf{0}) \cap \Sigma; \mathbf{P}^\bullet) \;\cong\; H^{s-k}(S_\varepsilon \cap \Sigma; \mathbb{Z}^\mu) = 0.$$

4. Proofs

Combining the results from the previous two sections, we find that, if the real link of the critical locus Σ at $\mathbf{0}$ is $(s-2)$-connected and $s \geq 3$, then we have for the Milnor fiber F of f at $\mathbf{0}$:

$$\widetilde{H}^{n-s}(F; \mathbb{Z}) \;\cong\; H^0(S_\varepsilon \cap \Sigma; \mathbb{Z}^\mu) \;\cong\; \mathbb{Z}^\mu;$$
$$\widetilde{H}^{n-k}(F; \mathbb{Z}) = 0, \text{ if } 2 \leq k \leq s-1.$$
$$\widetilde{H}^{k}(F; \mathbb{Z}) = 0, \text{ for } k \leq n-s-1, \text{ because of the result of [4]}$$
$$\widetilde{H}^{k}(F; \mathbb{Z}) = 0, \text{ for } k > n, \text{ because of the support condition.}$$

This proves the theorem.

Suppose now that, in addition to our other hypotheses, $s \geq 4$ and, for generic hyperplanes H, $S_\varepsilon \cap \Sigma \cap H$ is $(s-3)$-connected. Then, $f_{|_H}$ satisfies the hypotheses of the theorem, except that n is replaced by $n-1$ and s is replaced by $s-1$. Thus, for the Milnor fiber F_H:

$$\widetilde{H}^{n-s}(F_H; \mathbb{Z}) \;\;\cong\;\; \mathbb{Z}^\mu;$$
$$\widetilde{H}^{k}(F_H; \mathbb{Z}) = 0, \text{ if } k \neq n-2, n-1\ .$$

However, by the main result of [5], the Milnor fiber F is obtained from the Milnor fiber F_H by attaching cells in dimension n. Hence, $\widetilde{H}^{n-2}(F_H; \mathbb{Z}) \cong \widetilde{H}^{n-2}(F; \mathbb{Z})$, which we know is 0. This proves the corollary.

5. When the critical locus is an ICIS

Assume that the critical locus Σ of f is an isolated complete intersection singularity (ICIS) of dimension $s \geq 4$.

For an ICIS, the real link $S_\varepsilon \cap \Sigma$ is $(s-2)$-connected (see [8]). In addition, for a generic hyperplane H, the critical locus of $f_{|_H}$, which equals $\Sigma \cap H$, will also be an ICIS, but now of dimension $s-1$. Thus, $S_\varepsilon \cap \Sigma \cap H$ is $((s-1)-2)$-connected. Therefore, we are in the situation that we have considered above.

In his preprint [12] M. Shubladze asserts that if the singular locus Σ of f is a complete intersection with isolated singularity at $\mathbf{0}$ of dimension ≥ 3 and the Milnor number for transverse sections is 1 along $\Sigma \setminus \{\mathbf{0}\}$, the Milnor number of f at 0 has cohomology possibly $\neq 0$ only in dimensions 0, $n-s$ and n.

The results above show that, under the hypothesis of M. Shubladze, one obtains in a general way that the cohomology of the Milnor fiber of f at $\mathbf{0}$ is possibly $\neq 0$ in dimension 0, $n-s$, $n-1$ and n, and a similar result as the one of M. Shubladze in dimension 0, $n-s$, $n-1$ for the cohomology of the Milnor fiber of f restricted to a general hyperplane section if $\dim \Sigma \geq 4$.

Shubladze's result would follow immediately from our corollary, if it were true that every function such as that studied by Shubladze can be obtained as a generic hyperplane restriction of a function satisfying the same hypotheses. We cannot easily prove or disprove this result.

6. What if $S_\varepsilon \cap \Sigma$ is a homology sphere?

One might also wonder what happens if the real link of Σ is $(s-1)$-connected. This would, in fact, imply that $S_\varepsilon \cap \Sigma$ is a homology sphere. In this case, our earlier exact sequence immediately yields that $\widetilde{H}^{n-1}(F;\mathbb{Z}) = 0$.

A special case of $S_\varepsilon \cap \Sigma$ being a homology sphere would occur if Σ were smooth. However, in this case, when $s \geq 2$, Proposition 1.31 of [9] implies that the Milnor number cannot change at $\mathbf{0}$, i.e., we have a smooth μ-constant family, and so the non-zero cohomology of F occurs only in degrees 0 and $n-s$.

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